

WILLIAM FELLER AND TWENTIETH CENTURY PROBABILITY

1. Twentieth Century Probability

When William Feller was born in 1906, Lebesgue measure had just been invented, and Fréchet was to introduce measure on an abstract space about ten years later. Thus, the technical basis of modern mathematical probability was developed about the time of Feller's early childhood. Since that time the subject has been transformed, by no one more than by Feller himself, into an essential part of mathematics, contributing to other parts as well as drawing from them.

In the first part of our century, few probabilists felt comfortable about the basis of their subject, either as an applied or as a purely mathematical subject. In fact, it was commonly judged that there was no specific mathematical subject "probability," but only a physical phenomenon and a collection of mathematical problems suggested by this phenomenon. A probabilist joked that probability was "a number between 0 and 1 about which nothing else is known." In the discussions of the foundations of probability, there was no clear distinction made between the mathematical and the real. For example, one influential theory was that of von Mises, based on the concept of a "collective," which was defined as a sequence of observations with certain properties. Since "observation" is not a mathematical concept and since the properties were properties which no mathematical sequence could have, the theory could survive in its original form only by an affirmation that it was not a formal mathematical theory but an attempt at a direct description of reality. Instead, the theory was restricted to remove the mathematical objection, unfortunately losing in intuitive content what it gained in mathematical significance. The fate of the theory was an inevitable result of the increasing demand of mathematicians for exact definitions and formal rigor. The present formal correctness of mathematical probability only helps indirectly in analyzing real probabilistic phenomena. It is unnecessary to stress to statisticians that the relation between mathematics and these phenomena is still obscure. Or if not obscure it is clear to many but in mutually contradictory ways.

Formalizations of mathematical probability by Steinhaus in 1923 and Fréchet in 1930 were too incomplete to have much influence. The first acceptable formalization was by Kolmogorov in his 1933 monograph. Of course before that, and in fact for at least two centuries before that, there had been mathematicians who made correct and valuable contributions to mathematical probability. Mathematicians could manipulate equations inspired by events and expectations before these concepts were formalized mathematically as measurable sets and integrals. But deeper and subtler investigations had to wait until the blessing and curse of direct physical significance had been replaced by the bleak reliability of abstract mathematics.

Some probabilists have scorned the measure theory, functional analysis invasion of their subject, thinking it could do no more good than the discovery by Molière's character that he had been talking prose all his life. But in fact this invasion, to which Feller contributed so much, enriched the subject enormously in bringing it into the framework of modern mathematics, providing it with the possibility of undreamed of contacts with seemingly quite different mathematical fields. The definitive acceptance of mathematical probability as mathematics was, however, quite unnecessary for a large part of probabilistic research. For example, much of the distribution theory of sums of independent random variables can be considered an analysis of the convolutions of distribution functions. Random variables need never be mentioned. But even such researches are interesting largely because of their probabilistic significance. Many would not have been thought of and many more would have not been carried out, even if thought of, without this significance. Thus, the acceptance of probability as mathematics influenced research that could have been written without the probabilistic context. Even now this acceptance is not complete. In fact, many mathematics students are unaware of the place of probability in their subject. This situation is preserved by the special flavor given to probability by its linguistic heritage. Terms like "random variable" are here to stay and to continue misleading students on the state of probability theory, although "random variable" has a purely mathematical meaning whereas other familiar terms like "inclined plane" do not.

But the situation was even more confused forty years ago. A student can hardly visualize the difficulty of working in a field without a formal basis, without any sophisticated textbooks, in which it was respectable to have a serious discussion on what "really happens" when one tosses a coin infinitely often. The first sophisticated book was Lévy's remarkable 1937 book which was not written as a textbook and which yielded its treasures only to readers willing to make extreme efforts. A distinguished statistician in the early 1930's when asked how probability was taught at his university expressed surprise at the idea of teaching probability as a separate subject—it would be a "pointless *tour de force*." At that time random variables still were so mysterious that another distinguished mathematical statistician stated in a lecture that it was not known whether two random variables which were uncorrelated had to be independent. At that time the idea that a random variable was just (mathematically) a function was still so unfamiliar that it did not occur to the speaker to consider the sine and cosine functions on $(0, 2\pi)$, with the uniform distribution on that interval, as random variables, trivially uncorrelated and not independent.

But it is true that probability has lost some of its glamour along with its mystery. Luckily, the subject still has its basic physical background to draw on, still a source of ideas and problems. A further present feature is the interplay between mathematical probability and other parts of mathematics, for example, partial differential equations and potential theory.

It was a wonderful thing to be entering the field of probability when Feller did, in the early thirties. To one with his classical background, the field was obviously full of unsolved problems. Of course, it was not obvious at the time that the field was as rich as it has turned out to be, but it was clear that the subject was new in the sense that it had been barely touched by modern techniques. For example, discrete and continuous parameter Markov processes were just beginning to be studied in a nontrivial way. The multiplicity of classical type problems suggested by probability was such that it was not surprising that many probability papers indulged in probabilistic slang only long enough to reach the safe territory of integral equations or some other respectable established topic. Thereafter both writer and reader could relax, knowing that the introductory slang was as unessential as it was unexplained.

Wiener's work on Brownian motion (1924) was an exception. Although Wiener even later never used or knew the slang or even many of the elementary results of probability theory, his Brownian motion analysis was quite rigorous. In fact, an early problem in stochastic processes was to create a general theory which would include his approach to Brownian motion!

Since the thirties, mathematical probability has exploded. Several journals are devoted to it in its pure form and some other journals, for example the *Annals of Mathematical Statistics*, are barely distinguishable from probability journals. Sophisticated text books and specialized books are appearing all the time and there is even talk that the subject has reached or passed its peak. Feller, who was one of the researchers who brought the field to its present state, liked to relax in his advanced research by playing with elementary problems, polishing their known solutions. Let us hope that the new crop of researchers will be able to continue both his research and his purifying of old results.

2. William Feller

Feller was born in Zagreb, Yugoslavia on July 6, 1906, the ninth of twelve children of the well to do owner of a chemical factory. He attended the University of Zagreb (1923–1925), where he received the equivalent of an M.S. degree, and the University of Göttingen, where he received his Ph.D. in 1926 and remained until 1928. In 1928, he left Göttingen for the University of Kiel, where he worked as Privatdozent until in 1933 he refused to sign a Nazi oath and was forced to leave. It was at Kiel that he did his first work in probability. After a year (1933–1934) in Copenhagen, he went to Stockholm where he spent the next five years at the University and (July 27, 1938) married Clara Nielsen who had been his student in Kiel.

In 1939 the Fellers emigrated to Providence, where he became associate professor at Brown University and the first executive editor of *Mathematical Reviews*, founded that year. The only current mathematics review journal was then becoming corrupted by Nazi ideas. Much of the success of *Mathematical*

Reviews has been due to the policies initiated by Feller. *Mathematical Reviews* was founded in a less frantic scientific age when it was reasonable to have critical reviews, before the age of speed and preprints. There is now some opinion that traditional reviewing is an obsolete luxury, like peaceful universities. But even if this is true, the very speed of mathematical development that has made it true is in part a tribute to the success of *Mathematical Reviews* in furthering research.

In 1945, Feller accepted a professorship at Cornell University and remained there until 1950 when he moved to Princeton University as Eugene Higgins Professor of Mathematics. He held this position until his death (January 14, 1970), but in addition was a Permanent Visiting Professor at The Rockefeller University where he spent the academic years 1965–1966 and 1967–1968. A great attraction at The Rockefeller University was the opportunity to talk to geneticists.

Feller's first probability paper (1935) was on the central limit theorem, and in fact the properties of normalized sums of independent random variables were the subject of much of his later research, both from the point of view of distribution theory and from that of asymptotic bounds of the sums. Some of his deepest analytical work was in connection with the latter, work related to the general forms of the iterated logarithm law. It was in the context of distribution functions and their convolutions, not of random variables, and thus did not need the mathematical formalization of probability provided by Kolmogorov only a few years before. The central limit theorem paper gave necessary and sufficient conditions for convergence to a Gaussian limit.

In 1906, Markov did the first work on the sequences of random variables with the property that now bears his name. Progress was slow at first and some of Markov's work was repeatedly rediscovered. Kolmogorov's 1931 paper on continuous parameter Markov processes was a turning point, the first systematic investigation of these processes including the processes of diffusion. Feller wrote his first paper on these processes in 1936, going considerably beyond Kolmogorov and proving the appropriate existence and uniqueness theorems for the integrodifferential equations governing the transition probabilities. The main interest of both authors was in these equations. The stochastic processes themselves were secondary, although they inspired the analysis, and it is not surprising in view of the general historical remarks made above in Section 1 and the state of the subject at the time that Kolmogorov defined Markov processes incorrectly and Feller added an incorrect characterization (independent increments) to Kolmogorov's definition. All they needed was the Chapman-Kolmogorov equations, and the process giving rise to them was almost irrelevant. For Feller, as distinguished, say, from Lévy, it was usually the differential or integral equations or the semigroups arising in a probability context that interested him, rather than sample properties. On the other hand he kept these properties in mind, and although he usually did not treat them specifically he had a sure feeling for them and they inspired much of his analysis.

Feller completely transformed the subject of Markov processes. Going beyond

his 1935 paper, he put the analysis into a modern framework, applying semigroup theory to the semigroups generated by these processes. He observed that the appropriate boundary conditions for the parabolic differential equations governing the transition probabilities correspond on the one hand to the specification of the domains of the infinitesimal generators of the semigroups and on the other hand to the conduct of the process trajectories at the boundaries of the process state spaces. In particular, he found a beautiful perspicuous canonical form for the infinitesimal generator of a one dimensional diffusion. In this work, he was a pioneer yet frequently obtained definitive results.

Feller is best known outside the specialists in his field for his two volume work *An Introduction to Probability Theory and Its Applications*. He never tired of revising this book and took particular pleasure in finding new approaches, new applications, new examples, to improve it. The book is extraordinary for the almost bewildering multiplicity of its points of view and applications inside and outside pure mathematics. No other book even remotely resembles it in its combination of the purest mathematics together with a dazzling virtuosity of techniques and applications, all written in a style which displays the enthusiasm of the author. This style has made the book unexpectedly popular with non-specialists, just as its elegance and breadth, not to mention its originality, has made it an inspiration for specialists. Feller had planned two more volumes, and it would have been fascinating to see if his excitement in his subject could have brightened the usual dull measure theoretic details which would inevitably have had to appear in later volumes. Perhaps his unequalled classical background could have diluted, and made more palatable with applications and examples the concentrated dosage of preliminaries other mathematicians find necessary before studying Markov processes.

Feller was never heavily involved in statistics, although he was interested in it. He was not afraid of dirtying his fingers with numbers and in fact at one time he liked to work out least squares problems on hand computers as relaxation! He was president of the Institute of Mathematical Statistics in 1946. His attitude towards applications was unusual. On the one hand, his research was almost entirely in pure mathematics. On the other hand, he had far more than an amateur's interest in and knowledge of several applied fields, including statistics and genetics. He wrote a paper on extra sensory perception, and he wrote several papers applying the sophisticated ideas of a modern probabilist to genetics. He took an excited delight in applications of pure theory and nothing pleased him more than finding new ones. On the other hand he had a low boiling point for poor thinking, and nothing made him more excited than what he considered improper scientific thinking whether he favored or opposed the conclusion. Thus, he had great contempt for those who buttressed insufficient statistics on lung cancer and cigarettes with emotionalism or those who adduced uninformed arguments against Velikovsky's theories.

Feller was a member of the U.S. National Academy of Sciences, the Royal Danish Academy of Sciences, and the Yugoslav Academy of Sciences, as well as

a member of the American Academy of Arts and Sciences and the American Philosophical Society. His wife accepted the National Medal of Science for him shortly after his death. But apart from his mathematics those who knew him personally will remember Feller most for his gusto, the pleasure with which he met life, the excitement with which he drew on his endless fund of anecdotes about life and its absurdities, particularly the absurdities involving mathematics and mathematicians. To listen to him deliver a mathematics lecture was a unique experience. No one else could generate in himself as well as in his auditors so much intense excitement. In losing him, the world of mathematics has lost one of its strongest personalities as well as one of its strongest researchers.

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